SOME EXACT AND APPROXIMATE SOLUTIONS OF DYNAMIC PROBLEMS IN THE THEORY OF ELASTICITY

(O NEKOTORYKH TOCHNYKH I PRIBLIZHENNYKH RESHENIIAKH DINAMICHESKIKH ZADACH TEORII UPRUGOSTI)

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In much work which has appeared rectly, exact solutions of some dynamic problems have been replaced by exact solutions of more simple problems of the theory of elasticity, or by approximate solutions. In many cases the admissible error was not estimated and the possible region of applicability of the formulas obtained was not determined. In order to reduce the probability of erroneous conclusions to which this leads, it is useful to compare the results of exact and approximate estimates where possible.

An appreciable part of the previously completed investigations relates to wave processes in the neighborhood of the axes of symmetry in media with plane parallel boundaries. The smaller the edge effects in this neighborhood, the more complex the side and surface (Rayleigh) waves which appear in solid bodies far from the axes. Nevertheless with the boundary conditions encountered in practice, wave fields in different problems may differ appreciably from one another on the axes also.

In [1] the inadmissibility of calculations made according to formulas for acoustically approximate stresses inside a medium on the axes of symmetry has already been affirmed in the case of concentrated forces applied normal to the surface. The concentrated and distributed effects of the comparison were made of the calculated displacement and stress fields according to exact formulas for the semiplane and for unbounded media, according to acoustically approximate formulas, and according to formulas obtained on assumptions made in [2] (where the horizontal displacement was set equal to zero in the whole semispace, while the velocities of the transverse and longitudinal waves were set equal to one another).

Exact solutions of problems of oscillations of a semiplane under force applied at the boundaries have often been obtained in integral form by different methods. Nevertheless, because of a certain difficulty the formulas for a quantitative study of these solutions have not been provided completely enough in the case of a concentrated effect. Certain special results have been obtained only in certain cases of distributed effects. Upon taking into consideration that it is expedient (on the basis of economy of space and time) not to dwell on the derivation of known solutions, but rather to formulate the boundary and initial conditions for certain wave fields and to write out the formulas.

1. We introduce cylindrical coordinates r, θ, z, and on the boundary $z = 0$ of the semispace at rest $z \geqslant 0$ we have axially-symmetric stresses

$$\sigma_z = -\frac{n^2}{2\pi(1+n^2r^2)^{3/2}}\varepsilon(t), \qquad \tau_{rz} = 0 \tag{1.1}$$

where $\varepsilon(t) = 0$ for $t < 0$, $\varepsilon(t) = 1$ for $t > 0$, and n is a derived parameter [3].

By integration of the stresses (1.1) along the whole boundary it is not difficult to convince oneself that the summed effect corresponds to unit force. For $n \to \infty$ this force is concentrated at the origin. For small n the value of (1.1) changes slowly as a function of finite r, and, upon analyzing the wave picture near the boundaries around the origin, it is natural to expect that such effects are equivalent to uniformly distributed loads created by a simple plane wave. By giving different values to n, one may obtain all possible ranges of distributed effects and a higher or lower degree of approximation for the above mentioned limiting cases.

2. Upon putting the displacement components w and the horizontal q corresponding to the effect (1.1) into the formulas, and setting $r = 0$ (for which $q = 0$), we obtain on the axis of symmetry

$$w = -\frac{1}{4\pi^2\mu i}\int_l \left[\frac{g\alpha}{\zeta R(\zeta)\left[bt\zeta - z\alpha - \frac{1}{n}\right]} - \frac{2\alpha}{\zeta R(\zeta)\left[bt\zeta - z\beta - \frac{1}{n}\right]}\right] d\zeta \tag{2.1}$$

where the first summation describes the longitudinal waves and the second, the transverse waves: and

$$g = 2+\zeta^2, \quad \alpha = \sqrt{1+\gamma^2\zeta^2}, \quad \beta = \sqrt{1+\zeta^2}, \quad R(\zeta) = g^2 - 4\alpha\beta$$
$$\gamma = b / a, \qquad b = \sqrt{\mu/\rho}, \qquad a = \sqrt{(\lambda+2\mu)/\rho} \tag{2.2}$$

in which λ and μ denote the Lamé elastic constants, ρ the density, a and b the velocities of propagation of the longitudinal and transverse waves.

The contour l of integration passes so that the function summed under the integral sign on the right includes the pole, if the region is considered to be behind the wave front. For the region ahead of the front, the corresponding summation will not have a pole and the integral will reduce to zero.

If new dimensionless variables are introduced into Formula (2.1)

$$\tau = bt / z, \qquad \varkappa = 1 / nz \tag{2.3}$$

then after deformation of the contour l in the right semiplane and integration of the first and second sums corresponding to a circuit around the poles, with

$$\zeta_1 = \frac{\varkappa\tau + \sqrt{\tau^2 - \gamma^2 + \gamma^2\varkappa^2}}{\tau^2 - \gamma^2}, \qquad \zeta_2 = \frac{\varkappa\tau + \sqrt{\tau^2 - 1 + \varkappa^2}}{\tau^2 - 1} \tag{2.4}$$

appearing as roots of the equations $\tau\zeta - \alpha - \kappa = 0$, $\tau\zeta - \beta - \kappa = 0$, we obtain

$$w = \frac{1}{4\mu z}\left[\frac{g\alpha^2}{2R(\zeta)(1+\varkappa\alpha)}\bigg|_{\zeta=\zeta_1} \varepsilon(\tau-\gamma) - \frac{\alpha\beta}{R(\zeta)(1+\varkappa\beta)}\bigg|_{\zeta=\zeta_2} \varepsilon(\tau-1)\right] \tag{2.5}$$

The formulas for the stresses are found analogously as

$$\sigma_z = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left[\frac{g^2\alpha}{2\zeta R(\zeta)(1+\varkappa\alpha)}\bigg|_{\zeta=\zeta_1} \varepsilon(\tau-\gamma) - \frac{2\alpha\beta^2}{\zeta R(\zeta)(1+\varkappa\beta)}\bigg|_{\zeta=\zeta_2} \varepsilon(\tau-1)\right] \tag{2.6}$$

$$\sigma_r = \sigma_\theta = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left[\frac{[(1-2\gamma^2)\zeta^2 - 1]\,g\alpha}{2\zeta R(\zeta)(1+\varkappa\alpha)}\bigg|_{\zeta=\zeta_1} \varepsilon(\tau-\gamma) - \frac{\alpha\beta^2}{\zeta R(\zeta)(1+\varkappa\beta)}\bigg|_{\zeta=\zeta_2} \varepsilon(\tau-1)\right]$$

The solution for the concentrated effect comes from Formulas (2.4) to (2.6) by passing to the limit $\kappa \to 0$. This solution has been given before in [1,3].

3. We now consider solutions obtained on the first assumption of [2]. In this paper the horizontal displacements were set equal to zero in the whole semispace, and the vertical displacements were completely determined from known vertical forces on the boundary. No boundary conditions whatever are imposed as regards the shear stresses; they will be uniquely determined.

The problem in fact reduces to solution of the equation

$$\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial z^2} + b^2\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{3.1}$$

under the condition

$$(\lambda + 2\mu)\frac{\partial w}{\partial z}\bigg|_{z=0} = -\frac{n^2}{2\pi(1+n^2r^2)^{3/2}}\varepsilon(t) \tag{3.2}$$

Upon application of the method of incomplete separation of variables, the displacement w symmetrical about the axis may be expressed in integral form

$$w = -\frac{\gamma^2}{4\pi^2\mu i z}\int_l \frac{d\zeta}{\zeta\sqrt{1+\zeta^2}\,(\tau\zeta-\gamma\sqrt{1+\zeta^2}-\varkappa)} \tag{3.3}$$

From the theorem of residues we determine that

$$w = \frac{1}{\pi\mu z}\frac{\gamma}{2(\gamma+\varkappa\sqrt{1+\zeta_0^2})}\,\varepsilon(\tau-\gamma) \qquad \left(\zeta_0 = \frac{\tau\varkappa+\gamma\sqrt{\tau^2-\gamma^2+\varkappa^2}}{\tau^2-\gamma^2}\right) \tag{3.4}$$

Here ζ_0 is a root of the equation $\tau\zeta - \gamma\sqrt{(1+\zeta^2)} - \kappa = 0$.

We have for the stresses

$$\sigma_z = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left\{\frac{\sqrt{1+\zeta_0^2}}{2\zeta_0(\gamma+\varkappa\sqrt{1+\zeta_0^2})}\,\varepsilon(\tau-\gamma)\right\}, \qquad \sigma_r = \sigma_\theta = (1-2\gamma^2)\,\sigma_z \tag{3.5}$$

These give for $\kappa = 0$

$$w = \frac{1}{\pi\mu z}\frac{1}{2}\,\varepsilon(\tau-\gamma), \qquad \sigma_z = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left\{\frac{\tau}{2\gamma^2}\,\varepsilon(\tau-\gamma)\right\} \tag{3.6}$$

We note that the wave front corresponding to the solution of Equation (3.1) under the condition (3.2) for a concentrated effect ($\kappa = 0$) is an ellipsoid of rotation with the larger semiaxis at along the z-axis and with the smaller semiaxis bt along the r-axis.

4. If, in the semispace problem, the condition requiring the horizontal displacements to be zero is relaxed so that they are zero only on the boundary (the problem is now formulated correctly in distinction from the preceding case), then it is not difficult to convince oneself that a doubled solution of the problem is obtained for an unlimited medium [3]. The displacements and stresses corresponding to such a postulation are expressed on the axis of symmetry by the formulas

$$w = \frac{1}{\pi\mu z}\left[\frac{1+\gamma^2\zeta^2}{2\zeta^2(1+\varkappa\alpha)}\Big|_{\zeta=\zeta_1}\varepsilon(\tau-\gamma) - \frac{1}{2\zeta^2(1+\varkappa\beta)}\Big|_{\zeta=\zeta_2}\varepsilon(\tau-1)\right] \tag{4.1}$$

$$\sigma_z = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left[\frac{\alpha g}{2\zeta^3(1+\varkappa\alpha)}\Big|_{\zeta=\zeta_1}\varepsilon(\tau-\gamma) - \frac{\beta}{\zeta^3(1+\varkappa\beta)}\Big|_{\zeta=\zeta_2}\varepsilon(\tau-1)\right] \tag{4.2}$$

$$\sigma_r = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left[\frac{[(1-2\gamma^2)^2\zeta^2-1]\,\alpha}{2\zeta^3(1+\varkappa\alpha)}\Big|_{\zeta=\zeta_1}\varepsilon(\tau-\gamma) + \frac{\beta}{2\zeta^3(1+\varkappa\beta)}\Big|_{\zeta=\zeta_2}\varepsilon(\tau-1)\right] \tag{4.3}$$

in which ζ_1 and ζ_2 have the values as in (2.4). We obtain for $\kappa = 0$

$$w = \frac{1}{\pi\mu z}\left[\frac{\tau^2}{2}\,\varepsilon(\tau-\gamma) - \left(\frac{\tau^2}{2}-\frac{1}{2}\right)\varepsilon(\tau-1)\right] \tag{4.4}$$

$$\sigma_z = -\frac{1}{\pi z^2}\frac{\partial}{\partial \tau}\left[\left(\tau^3 + \frac{1-2\gamma^2}{2}\tau\right)\varepsilon(\tau-\gamma) - (\tau^3-\tau)\,\varepsilon(\tau-1)\right] \tag{4.5}$$

$$\sigma_r = -\frac{1}{\pi z^2}\frac{\partial}{\partial \tau}\left[\left(-\frac{\tau^3}{2} + \frac{1-\gamma^2}{2}\tau\right)\varepsilon(\tau-\gamma) + \left(\frac{\tau^3}{2} - \frac{\tau}{2}\right)\varepsilon(\tau-1)\right] \tag{4.6}$$

5. The solution of the acoustical problem may be found from the solution of the dynamic problem by passing to the limit and requiring the velocity of the transverse waves to approach zero. Actually, upon rejecting the second sums corresponding to transverse waves in Formulas (2.5) and (2.7), we replace b by bu in the first summations (equivalent to replacement of $\mu = \rho b^2$ by μu^2, γ by γu, and ζ_1 by $\zeta_1 u^{-1}$), and let u approach zero considering b (and also μ) as invariable parameters. We find as a result

$$w = \frac{1}{\pi\mu z}\frac{\alpha^2}{2\zeta^2(1+\varkappa\alpha)}\bigg|_{\zeta_1}\varepsilon(\tau-\gamma), \qquad \sigma_z = \sigma_r = \sigma_\theta = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left[\frac{\alpha}{2\zeta(1+\varkappa\alpha)}\bigg|_{\zeta_1}\varepsilon(\tau-\gamma)\right] \tag{5.1}$$

Formulas (5.1) take for $\kappa = 0$ the form

$$w = \frac{1}{\pi\mu z}\frac{\tau^2}{2}\,\varepsilon(\tau-\gamma), \qquad \sigma_z = \sigma_r = \sigma_\theta = -\frac{1}{\pi z^2}\frac{\partial}{\partial\tau}\left(\frac{\tau}{2}\,\varepsilon(\tau-\gamma)\right) \tag{5.2}$$

One can be easily convinced that Formulas (4.1) to (4.6) lead to the same results on passing to the limit. Analogously, passing to the limit in Formulas (3.4) gives the values

$$w = \frac{1}{\pi\mu z}\frac{\gamma(\tau-\gamma)}{2\varkappa^2}\,\varepsilon(\tau-\gamma)$$
$$\sigma_z = -\frac{1}{\pi z^2}\frac{1}{2\varkappa^2}\,\varepsilon(\tau-\gamma), \qquad \sigma_r = \sigma_\theta = (1-2\gamma^2)\,\sigma_z \tag{5.3}$$

reducing (from an incorrect formulation of the problem) to infinity upon passing to the concentrated effect (no physical significance is attached to the displacements in the acoustical case since they grow without limit with increasing τ).

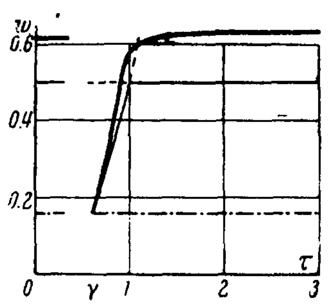

Fig. 1.

6. In [2], in addition to the assumption of zero horizontal displacements, it was considered possible also to equate the velocities of propagation of the transverse and longitudinal waves. Formula (3.3) gives in this case

$$w = \frac{1}{\pi\mu z}\frac{\gamma^2}{2(1+\varkappa\alpha)}\bigg|_{\zeta=\zeta_1}\varepsilon(\tau-\gamma) \tag{6.1}$$

coinciding with the acoustical solution for $\tau = \gamma$ (on the longitudinal wave front), and for $\tau > \gamma$ it grows monotonically up to a certain limit but remains smaller than the previously mentioned solution. With these assumptions the stress σ_z coincides with the acoustical solution for any τ, but $\sigma_r = \sigma_\theta$ takes the paradoxical value $- \sigma_z$. In fact, the assertion of the author of [2] that solutions of nonacoustical problems are obtained from the two assumptions for solid media is shown to be untrue for the stress σ_z. In general his solution (considering also the σ_r, σ_θ components) is worse than the acoustical solution or any of the other solutions considered above which reflect the actual picture.

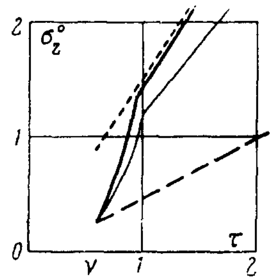

Fig. 2.

7. We present results of quantitative estimates from solutions of different problems with different distributed forces on the boundary $z = 0$.

First we note that for large values of κ where the boundary forces near the axes of symmetry are close to a uniformly distributed dynamic load but for finite times τ including the arrival times of the longitudinal and transverse wave fronts, the main part of the dynamic process is described as a plane longitudinal wave. Asymptotic estimates of this are given by Formulas (5.3) for all problems considered. Only for $\sigma_r = \sigma_\theta$ in the acoustical case (when $\gamma = 0$) we have in place of the equality $\sigma_r = \sigma_\theta = (1 - \gamma^2)\sigma_z$ the natural equality $\sigma_r = \sigma_\theta = \sigma_z$; in the case of the arbitrary assumption of equality of wave velocities we obtain the physically inadmissible equality $\sigma_r = \sigma_\theta = - \sigma_z$.

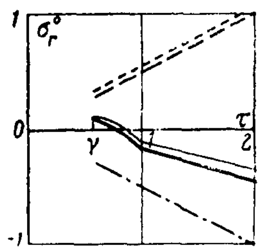

Fig. 3.

For a concentrated effect when $\kappa = 0$, the displacement w at a point on the axis of symmetry for $\gamma^2 = 0.33$ is described in Fig. 1 with accuracy up to a multiplier $1/\pi\mu z$. In this figure, as well as in all succeeding figures, the continuous thick line relates to the solution for a solid semispace, the thin line to the doubled solution for a solid unlimited medium, the dashed line to the acoustical problem, the line of points to the solution obtained on the assumption of zero horizontal displacements over the whole semispace, and the dot-dash line relates to solutions obtained on the basis of the two assumptions made in [2] ($q = 0$, equality of velocities of longitudinal and transverse waves). The corresponding stresses $\sigma_z^{\,o}$ and $\sigma_r^{\,o} = \sigma_\theta^{\,o}$ are shown in Figs. 2 and 3, Within the accuracy of a multiplier $1/\pi z^2$. We note that in Fig. 1 the dotted line and the dashed line coincide in certain portions with the thin continuous line, and that in Fig. 2 the line of points coincides with the dashed line.

Estimates of the wave fields for $\kappa = 1$ are also presented. The force distribution at the boundary $z = 0$ (disregarding the factor $1/2\pi z^2$) is shown in Fig. 4 as a function of r/z. The corresponding displacements are shown in Fig. 5, the stress σ_z° in Fig. 6, and the stresses $\sigma_r^{\circ} = \sigma_\theta^{\circ}$ ιν in Fig. 7, to the same accuracy as before. (The dot-dash line coincides with the dashed line in Fig. 6).

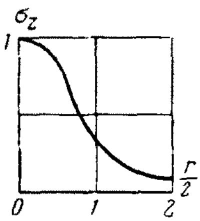

Fig. 4.

8. Comparison of the curves shows that the wave field in the solid semispace on the axis of symmetry is described best of all by the doubled solution for an unlimited medium. The displacements w and the stress σ_z are also well described by the solution obtained on the assumption of zero horizontal displacements q over the whole semispace. The solution of the acoustical problem, as well as the solution obtained in [2] on the two previously mentioned assumptions gives values too low (and also untrue for certain components) even for values of κ appreciably different from zero.

This same regularity also appears upon passing from the $\varepsilon(t)$ effect to an arbitrary effect $f(t)$, which for ease of calculation is suitably approximated by a broken line [1]. If the duration of the effect (or more correctly, its most rapidly decaying portion) exceeds the difference in arrival time of the longitudinal and the transverse waves (always true at points near the source), then as the curves show, the wave field behind the transverse wave front begins to play an essential or oven a principal role in the dynamic process. This field approximates a static field for the $\varepsilon(t)$ effect, while the $f(t)$ effect describes processes which, as we understand them, must be considered as being close to quasistatic oscillations. For protracted and sufficiently smooth $f(t)$ effects, the sharpness of the change in the field near the fronts, essential in the $\varepsilon(t)$ effect, is shown to be insignificantly small and the process is established as being completely quasistatic.

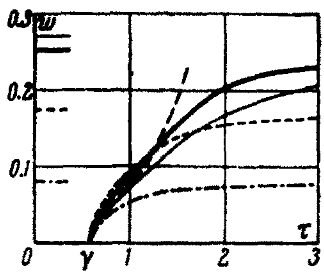

Fig. 5.

The sharp variation in the wave field in the prefrontal region plays an essential role in linear problems of the theory of elasticity only

$$M_x^{(s)} = -\frac{2Eh^3}{3(1-\nu^2)}\left(\frac{\partial^2 w^{(s)}}{\partial x^2} + \nu\frac{\partial^2 w^{(s)}}{\partial y^2}\right) \quad (xy), \qquad H^{(s)} = -\frac{2Eh^3}{3(1+\nu)}\frac{\partial^2 w^{(s)}}{\partial x \partial y}$$

$$V_x^{(s)} = \frac{\partial M_x^{(s)}}{\partial x} + \frac{\partial H^{(s)}}{\partial y} - h^{s-1}\int_{-1}^{+1} \tau_{xz}^{*(s)}\, d\zeta \quad (xy) \tag{16.3}$$

$$\frac{\partial^2 M_x^{(s)}}{\partial x^2} + \frac{\partial^2 H^{(s)}}{\partial x\, \partial y} + \frac{\partial^2 M_y^{(s)}}{\partial y^2} = 4h^{s-1}\sigma_{z3}^{(s)}$$

In making the sth approximation the quantities with an asterisk and a superscript $(s - 1)$ can be taken as known. In addition, taking into account formulas (3.6), we can also consider $\tau_{xz}^{*(s)}$, $\tau_{yz}^{*(s)}$ and $\sigma_{z3}^{(s)}$ in (16.3) as known quantities. Thus Expressions (16.3) are the equations of the classical theory of plates. In these expressions the terms containing $\tau_{xz}^{*(s)}$, $\tau_{yz}^{*(s)}$ and $\sigma_{z3}^{(s)}$ represent the externally applied forces and moments respectively.

For $s = 1$ we have that $\tau_{xz}^{*(s)} = 0$, $\tau_{yz}^{*(s)} = 0$, $4\sigma_{z3}^{(s)} = p$ and the conditional load applied to the plate coincides with that considered in an analysis based on the classical theory.

For $s = 2$ we have that $\tau_{xz}^{*(s)} = 0$, $\tau_{yz}^{*(s)} = 0$, $\sigma_{z3}^{*(s)} = 0$ and the conditional load vanishes. For $s > 2$ the conditional forces and moments in the sth approximation depend on the stresses of the $(s - 2)$th approximation. The conditional load intensity diminishes as s increases according to the law h^{s-1}.

The boundary conditions required for finding the biharmonic function $B^{(1)}$ are given by (15.2) to (15.4) and (15.6). In these equalities $\tau_1^{(1)}$ can be expressed according to Formula (12.3) and $u_1^{(1)}$ can be written in terms of $w_0^{(1)}$ with the aid of (3.3). Taking this into account and making use of Formulas (16.2), we obtain:

the boundary conditions corresponding to a free edge

$$M_x^{(1)} = 0, \qquad V_x^{(1)} + \partial H^{(1)} / \partial y = 0$$

the boundary conditions corresponding to a fully fixed edge

$$\partial w^{(1)} / \partial x = 0, \qquad w^{(1)} = 0$$

and the boundary conditions corresponding to a simply-supported edge

$$M_x^{(1)} = 0, \qquad w^{(1)} = 0$$

It follows that the first approximation of the basic iteration process is equivalent to the classical theory of plates, in the identity